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The scalars of $N = 2$, $D = 5$ and attractor equations.

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Abstract

Theories in 5 dimensions with minimal supersymmetry are studied for domain-wall solutions and in the context of the AdS/CFT correspondence. The scalar manifold is a product of a very special real manifold and a quaternionic-Kähler manifold. Superconformal methods can clarify the structure of these manifolds, which are part of the family of special manifolds. BPS solutions depending on the scalars and a warp factor of the 5-dimensional metric with a flat 4-dimensional metric can interpolate between critical points determined by algebraic attractor equations. The mixing of vector and hypermultiplets is essential to obtain UV and IR critical points.

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1 Introduction

Supersymmetric theories in 5 dimensions have got new interest in the context of the AdS/CFT correspondence and for a supersymmetrisation of the Randall–Sundrum (RS) scenario. In both cases one uses at the end a metric of the form

$$ds^2 = a(x^5)^2 dx^\mu dx^\nu \eta_{\mu\nu} + (dx^5)^2, \quad (1.1)$$

where $\underline{\mu}, \underline{\nu} = 0, 1, 2, 3$. We thus have a flat 4-dimensional space with a warp factor a that depends on the fifth direction x^5 . We first want to draw the attention on two different concepts, which are called either ‘smooth solutions’ or ‘singular sources’. With ‘smooth solutions’ we [1] mean that we consider the generic 5-dimensional supergravity theory [2], and we look for a solution where the warp factor has the required form as explained above. On the other hand, ‘singular sources’ means that 3-brane sources are inserted at specific places. The fifth dimension is in that case an orbifold S^1/\mathbb{Z}_2 , and the sources sit at its fixed points. The 5-dimensional bulk action is supplemented by a brane term that involves delta functions $\delta(x^5 - x_{\text{fixed}}^5)$ times a four-dimensional action. We [3] provided a simple general mechanism how to implement supersymmetry in such a scenario despite the singularities in spacetime due to the delta functions. It introduces a new 4-form field in the bulk supergravity, that appears also as 4-form for the brane Wess–Zumino term. This construction has been reviewed also in [4] and we will omit therefore this part of the talk from these proceedings. The mechanism inspired a generalization to 10 dimensions, leading to new formulations of type IIA (and also type IIB) supergravity in [5], as reviewed in the talk of Eric Bergshoeff in this school. This setup is similar to the Hořava–Witten [6] theory, which was reduced to 5 dimensions in [7]. It has been considered by various groups [8, 9, 10, 11], and our bulk&brane solution was used for cosmology in [12, 13].

We restrict ourselves here to ‘smooth solutions’. The main part of this review will be devoted to the structure of the manifolds of scalars that appear in these theories. We explain how the superconformal methods clarify the structure, referring to new results of [14, 15], and pay special attention to definitions of quaternionic(-Kähler) manifolds. At the end, we consider the scalar-dependent solutions in the warped background (1.1), and show how the critical points are determined by algebraic attractor equations [1], generalizing earlier similar equations with vector multiplets in 4 dimensions [16, 17, 18] to include hypermultiplets. To analyse the properties of the critical points, a general formula on the scalar mass matrix [19] gives a lot of insight.

2 $N = 2$, $D = 5$ supergravity and its scalars

Pure $N = 2$, $D = 5$ supergravity¹ contains a graviton, two gravitini and a graviphoton (spin 1). The theory can contain vector multiplets, each containing a vector, a doublet of spinors and a scalar. These scalars define a ‘very special real manifold’, as we will explain. Furthermore, there are hypermultiplets each containing 2 spinors and 4 scalars. The latter define a quaternionic-Kähler manifold. Note that we will not consider tensor multiplets. The

¹ $N = 2$ means 8 real supercharges: 4-component spinors in an $SU(2)$ doublet. It is minimal supersymmetry in 5-dimensional Minkowski space. Note that for other signatures (2 or 3 time directions) one can impose Majorana conditions such that only 4 of them survive ($N = 1$).

antisymmetric tensors are dual to vectors in 5 dimensions. That duality is only for Abelian couplings, but that is all that we consider here. Remark, however, that for non-Abelian theories, antisymmetric tensor multiplets lead to more general possibilities, as discussed in detail in [20].

We first review the superconformal construction of matter couplings in supergravity. We then discuss the two scalar manifolds. Finally we discuss the consequences of gauging isometries of the quaternionic manifold using the vector fields.

2.1 Superconformal tensor calculus

Superconformal tensor calculus provides a way to construct matter-coupled super-Poincaré theories. The aim here is thus not to end up with a superconformal theory, but rather to provide a way to construct the general supergravity theories. Using conformal invariance facilitates the construction of the theory and leads to more insight in the structure of the theory. E.g. the structure of special Kähler manifolds was developed by using superconformal tensor calculus.

The method thus consists of first constructing a theory invariant under the superconformal group, and then fixing all the invariances that are not required for a super-Poincaré invariant theory. A superconformal group consists of the conformal group (with translations P_μ , Lorentz rotations $M_{\mu\nu}$, dilatations D , and special conformal generators K_μ), supersymmetry Q , and a special supersymmetry S , and finally some R -symmetry. The latter is a bosonic group that acts on the supersymmetries and appears in the anticommutator of ordinary and special supersymmetry. In our case, this group is $SU(2)$, and the full superconformal algebra defines the supergroup $F^2(4)$ (where the index '2' indicates the particular real form of the superalgebra, see table 5 in [21]). Its bosonic subgroup is $SO(5, 2) \times SU(2)$, where the first factor is the conformal group and the second is the R -symmetry group.

Note that in this case the bosonic subgroup is a direct product of the conformal group and the R -symmetry group. This is the case in the superconformal algebras classified by Nahm [22]. It implies that bosonic symmetries that are not in the conformal algebra are spacetime scalars. This is not a necessity. Other examples have been considered first in 10 and 11 dimensions in [23]. Recently, a new classification has appeared in [24] of which we can extract table 1 for dimensions from 3 to 11. The bosonic subgroup contains always two factors. One contains the conformal group. If that factor is really the conformal group, then the algebra appears in Nahm's classification. Note that 5 dimensions is a special case. There is a generic superconformal algebra for any extension. But for the case $N = 2$ there exists a smaller superconformal algebra that is in Nahm's list. So far, superconformal tensor calculus has only been based on algebras of Nahm's type. Note that for $D = 6$ or $D = 10$, where one can have chiral spinors, only the case that all supersymmetries have the same chirality has been included.

For the methods that are used in superconformal tensor calculus, we refer to existing reviews, as the one that appeared in the Karpacz proceedings of 1983 [25]. A useful example is its application in $N = 1$, $D = 4$ supergravity [26, 27], that has been written down in detail in [28].

The basic multiplet is the one that contains the gauge field of all the symmetries in the superconformal group. This is called the Weyl multiplet, and has recently been constructed for $N = 2$, $D = 5$ in [14, 15]. There are two versions, as it is the case in six dimensions [29].

D	supergroup	bosonic group	
3	$OSp(N 4)$	$Sp(4) = SO(3, 2)$	$SO(N)$
4	$SU(2, 2 N)$	$SU(2, 2) = SO(4, 2)$	$SU(N) \times U(1)$
5	$OSp(8^* N)$	$SO^*(8) \supset SO(5, 2)$	$USp(N)$
	$F(4)$	$SO(5, 2)$	$SU(2)$
6	$OSp(8^* N)$	$SO^*(8) = SO(6, 2)$	$USp(N)$
7	$OSp(16^* N)$	$SO^*(16) \supset SO(7, 2)$	$USp(N)$
8	$SU(8, 8 N)$	$SU(8 8) \supset SO(8, 2)$	$SU(N) \times U(1)$
9	$OSp(N 32)$	$Sp(32) \supset SO(9, 2)$	$SO(N)$
10	$OSp(N 32)$	$Sp(32) \supset SO(10, 2)$	$SO(N)$
11	$OSp(N 64)$	$Sp(64) \supset SO(11, 2)$	$SO(N)$

Table 1: *Superconformal algebras, with the two parts of the bosonic subalgebra: one that contains the conformal algebra and the other one is the R-symmetry. In the cases $D = 4$ and $D = 8$, the $U(1)$ factor in the R-symmetry group can be omitted for $N \neq 4$ and $N \neq 16$, respectively.*

(gauge) field	$D = 4$	$D = 5$	$D = 6$	gauge transf.	subtracted
e_μ^a	5	9	14	P^a	M_{ab}, D
b_μ	compensating K^a			D	K^a
ω_μ^{ab}	composite			M^{ab}	
f_μ^a	composite			K^a	
$V_{\mu i}^j$	9	12	15	$SU(2)$	
A_μ	3	—	—	$U(1)$	
ψ_μ^i	16	24	32	Q^i	S^i
ϕ_μ^i	composite			S^i	
T_{ab}, T_{abc}^-	6	10	10		
D	1	1	1		
χ^i	8	8	8		
TOTAL	24 + 24	32 + 32	40 + 40		

Table 2: *Standard Weyl multiplet in 4, 5 and 6 dimensions.*

Both versions have 32+32 components and are equivalent. In fact, there is a procedure to go from one to the other [14]. We will restrict ourselves to one version, which is the one used primarily also in 4 and 6 dimensions. Its content is given in table 2. We indicate for each field the number of components in each dimension, the symmetry for which it is a gauge field, and possibly other gauge transformations that have been used to reduce its number of degrees of freedom in this counting.

Once one has this multiplet, one can add other multiplets, i.e. representations of the superconformal algebra. In order to satisfy this algebra, the transformation laws of the fields in these multiplets will involve the fields of the Weyl multiplet. Then one constructs a superconformal invariant action, and finally one has to fix the superfluous symmetries. Remark that we already used the K^a symmetry to put the gauge field of dilatations, b_μ , equal to zero. The remaining superfluous symmetries are therefore the dilatations D , the special supersymmetries S^i , and the R-symmetry $SU(2)$.

spin	pure SG	vector mult.	hypermult.	field	indices
2	1			e_μ^a	$\mu, a = 0, \dots, 4$
$\frac{3}{2}$	2			ψ_μ^i	$i = 1, 2$
1	1	n		A_μ^I	$I = 0, \dots, n$
$\frac{1}{2}$		$2n$	$2r$	λ_i^x, ζ^A	$A = 1, \dots, 2r$
0		n	$4r$	ϕ^x, q^X	$x = 1, \dots, n; X = 1, \dots, 4r$

Table 3: *Multiplets and fields of the super-Poincaré theories*

spin	Weyl	vector	hyper	gauge fix	auxiliary
2	e_μ^a				
$\frac{3}{2}$	ψ_μ^i				
	$V_{\mu i}^j, T_{ab}$				auxiliary
1		$n + 1$			
$\frac{1}{2}$	χ^i	$2(n + 1)$	$2(r + 1)$	2: S	χ^i with 2 others
0	D	$n + 1$	$4(r + 1)$	1: dilatations, 3: $SU(2)$	D and 1 other

Table 4: *Multiplets and fields in the superconformal construction*

As we mentioned in the beginning of this chapter, we consider general couplings with n vector multiplets and r scalar multiplets. Table 3 gives their content, the names that we use for the fields, and the corresponding range of indices. In the superconformal method, these are obtained in a different way. One starts with the Weyl multiplet, and adds vector multiplets and hypermultiplets in representations of the superconformal algebra. As well for the vector multiplets as for the hypermultiplets, one starts by adding one more multiplet than appears in the final super-Poincaré theory. These ‘compensating multiplets’ contain the degrees of freedom that will be gauge-fixed. This is schematically represented in table 4. It is indicated how the superfluous symmetries are fixed, and how some of the fields of the Weyl multiplet serve as Lagrange multipliers eliminating degrees of freedom of the spin 1/2 and scalar fields. The field $V_{\mu i}^j$ will be eliminated by its field equation, and will play the role of $SU(2)$ curvature of the quaternionic manifold defined by the hyperscalars. The field T_{ab} will become a function of the field strengths of the vectors in the vector multiplet (dressed by the scalars), and plays the role of gauge field that enters in the gravitino transformation (related to the central charge).

2.2 Very special real and quaternionic-Kähler manifolds

The manifolds of supergravity-matter couplings in $D = 5$ are similar to those that are known from $N = 2$ in 4 dimensions. Table 3 would be nearly identical for 4 dimensions, except that each vector multiplet then contains two scalars. The supersymmetry defines a complex structure, and the manifold is Kählerian. In $N = 1$ supergravity, general Kähler manifolds are possible. In $N = 2$ they are restricted to a category that is called ‘special Kähler manifolds’ [30]. The quartets of scalars in hypermultiplets are connected by 3 complex structures and the manifold is quaternionic-Kähler [31]. Another recent review containing the fundamental facts of these manifolds is given in [32].

2.2.1 Very special real manifolds

We first consider the vector multiplets [33]. In 5 dimensions, these have real scalars (one of the scalars of 4 dimensions sits in the $5d$ -vector). We define 'very special real manifolds' [34] as those that appear in these couplings of vector multiplets to 5-dimensional supergravity. It is clear from the above, that they can be described in superconformal tensor calculus by starting with $n + 1$ scalars, which we denote h^I , as in table 4. Then we impose a dilatational gauge choice. This defines an n -dimensional hypersurface in the $(n + 1)$ -dimensional space.

The locally supersymmetric action of the vector multiplets in 5 dimensions contains always a Chern–Simons term of the form $C_{IJK} A^I dA^J dA^K$. In order for this to be gauge-invariant, the C_{IJK} have to be constant. This tensor is completely symmetric in its indices, and supersymmetry implies that the full action is determined by these constants (up to the choice of coordinates on the manifold). Thus the set of numbers C_{IJK} are all one needs to specify a very special real manifold [33]. For an arbitrary set, one still has to verify whether they allow a non-empty domain with positive-definite metric on the scalar manifold.

The dilatational gauge choice that is most appropriate is the condition

$$C_{IJK} h^I(\phi) h^J(\phi) h^K(\phi) = 1. \quad (2.1)$$

ϕ^x are coordinates on this manifold such that the embedding $h(\phi)$ satisfies the condition. The metric on the scalar manifold is then

$$g_{xy} = -3(\partial_x h^I)(\partial_y h^J) C_{IJK} h^K. \quad (2.2)$$

2.2.2 Quaternionic-Kähler manifolds

Let us now look at the other side: the hypermultiplets. We first define quaternionic manifolds. We start with a $4r$ -dimensional manifold with coordinates q^X . At each point there is a tangent space where the vectors are labelled with indices (iA) (see ranges in table 3). These are connected by $4r \times 4r$ vielbeins f_X^{iA} or their inverses f_{iA}^X . We will here introduce the quaternionic manifolds starting from these vielbeins. Quaternionic manifolds entered physics in [31], and [35] contains a lot of interesting properties. There were two workshops on quaternionic geometry where mathematics and physics results were brought together [36, 37]. Other recent papers that review the properties of quaternionic manifolds are [32, 38].

For supersymmetry, starting from vielbeins is a convenient approach because these are the objects that one uses from the very beginning, i.e. in the supersymmetry transformations of the hyperscalars:

$$\delta(\epsilon) q^X = f_{iA}^X \epsilon^i \zeta^A. \quad (2.3)$$

Almost quaternionic manifolds We thus have

$$f_Y^{iA} f_{iA}^X = \delta_Y^X, \quad f_X^{iA} f_{jB}^X = \delta_j^i \delta_B^A. \quad (2.4)$$

These vielbeins satisfy a reality condition defined by matrices E_i^j and ρ_A^B that satisfy

$$E E^* = -\mathbb{1}_2, \quad \rho \rho^* = -\mathbb{1}_{2r}. \quad (2.5)$$

One may choose a standard antisymmetric form for ρ and identify E with ε by a choice of basis. The reality condition for the vielbeins are

$$(f_X^{iA})^* = f_X^{jB} E_j^i \rho_B^A. \quad (2.6)$$

The transformations on variables with an A index are by the reality condition restricted to $Gl(r, Q) = SU^*(2n) \times U(1)$.

We define complex structures as ($r = 1, 2, 3$ and using the three sigma matrices)

$$J_X^{Yr} \equiv -i f_X^{iA} (\sigma^r)_i^j f_{jA}^Y, \quad \Rightarrow \quad J_X^Y{}_i{}^j \equiv i J_X^{Yr} (\sigma^r)_i^j = 2 f_X^{jA} f_{iA}^Y - \delta_i^j \delta_X^Y. \quad (2.7)$$

We use the same transition between triplet and doublet notation below for other quantities. The complex structures satisfy, due to (2.4), the quaternion algebra

$$J^r J^s = -\mathbb{1}_{4r} \delta^{rs} + \varepsilon^{rst} J^t. \quad (2.8)$$

This defines the manifold to be ‘almost quaternionic’.

Quaternionic manifolds We now suppose that there is a torsionless connection $\Gamma_{XY}^Z = \Gamma_{YX}^Z$. Consider then

$$\Omega_{XjB}{}^{iA} \equiv f_{jB}^Y (\partial_X f_Y^{iA} - \Gamma_{XY}^Z f_Z^{iA}) = -\omega_{Xj}{}^i \delta_B^A - \omega_{XB}{}^A \delta_j^i, \quad (2.9)$$

where $\omega_{Xj}{}^i$ is traceless. If this $\Omega_{XjB}{}^{iA}$, for each X , would be a general $4r \times 4r$ matrix, then we would say that the holonomy is not restricted (or sits in $Gl(4r)$). The splitting as in the right-hand side of this equation implies that the holonomy group is restricted to $SU(2) \times Gl(r, Q)$. We can write (2.9) as the covariant constancy of the vielbein:

$$\partial_X f_Y^{iA} - \Gamma_{XY}^Z f_Z^{iA} + f_Y^{jA} \omega_{Xj}{}^i + f_Y^{iB} \omega_{XB}{}^A = 0, \quad (2.10)$$

with composite gauge fields for $SU(2)$ and $Gl(r, Q)$. These conditions promote the almost quaternionic structure to a quaternionic structure, and the manifold is ‘quaternionic’. If the $SU(2)$ connection is zero, they are called ‘hypercomplex’.

The integrability condition of (2.10), (multiplied by a vielbein) is

$$R^Z{}_{WXY} = f_{iA}^Z f_W^{jA} \mathcal{R}_{XYj}{}^i + f_{iA}^Z f_W^{iB} \mathcal{R}_{XYB}{}^A = -J_W^{Zr} \mathcal{R}_{XY}{}^r + f_{iA}^Z f_W^{iB} \mathcal{R}_{XYB}{}^A, \quad (2.11)$$

where respectively the metric curvature $R^Z{}_{WXY} \equiv 2\partial_{[X} \Gamma_{Y]W}^Z + 2\Gamma_{V[X}^Z \Gamma_{Y]W}^V$, the $SU(2)$ curvature $\mathcal{R}_{XY}{}^r \equiv 2\partial_{[X} \omega_{Y]}{}^r + 2\omega_{[X}{}^s \omega_{Y]}{}^t \varepsilon^{rst}$, and the $Gl(r, Q)$ curvature $\mathcal{R}_{WXB}{}^A$ appear.

Quaternionic-Kähler manifolds Quaternionic-Kähler manifolds (including ‘hyperkähler’ for the case that the $SU(2)$ curvature vanishes) by definition have a metric. First define an Hermitian metric d_{AB} such that $C_{AB} \equiv \rho_A^C d_{CB} = -C_{BA}$. By redefinitions of the basis [39] one may diagonalize d while simultaneously bringing C to a canonical form

$$d = \begin{pmatrix} \mathbb{1}_{2p} & 0 \\ 0 & -\mathbb{1}_{2(r-p)} \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -1 & 0 \\ & & & \ddots \end{pmatrix}. \quad (2.12)$$

The subgroup of $G\ell(r, Q)$ that preserves d is $USp(2p, 2r - 2p)$. We define then the metric of the manifold to be²

$$g_{XY} = f_X^{iA} C_{AB} \varepsilon_{ij} f_Y^{jB}. \quad (2.13)$$

We use ε_{ij} and C_{AB} to raise and lower indices according to the NW–SE convention

$$A_i = A^j \varepsilon_{ji}, \quad A^i = \varepsilon^{ij} A_j, \quad A_A = A^B C_{BA}, \quad A^A = C^{AB} A_B. \quad (2.14)$$

One can check that this raising and lowering of indices, together with the usual raising and lowering of indices X by g_{XY} and its inverse is consistent with defining f_X^{iA} and f_{iA}^X as each others inverses.

The choice of the signature in (2.12) is relevant for the reality conditions, which are then $(f_X^{iA})^* d_{AB} = f_{X i B}$. Our notations hide d from other places.

For $r > 1$ one can prove that these manifolds are Einstein, and that the $SU(2)$ curvatures are proportional to the complex structures:

$$R_{XY} = \frac{1}{4r} g_{XY} R, \quad \mathcal{R}_{XY}^r = \frac{1}{2} \nu J_{XY}^r, \quad \nu = \frac{1}{4r(r+2)} R. \quad (2.15)$$

(with $R_{XY} = R^Z_{XZY}$). For $r = 1$ this is part of the definition of quaternionic-Kähler manifolds. Hyperkähler manifolds are those where the $SU(2)$ curvature is zero, and these are thus also Ricci-flat.

Supergravity In supergravity we find all these constraints from requiring a supersymmetric action. Moreover, we need for the invariance of the action that the last equation of (2.15) is satisfied with $\nu = -1$. This implies that the scalar curvature is $R = -4r(r+2)$. This excludes e.g. the compact symmetric spaces.

2.2.3 The family of special manifolds

We now place these manifolds in the context of the manifolds that are obtained for supersymmetries with 8 real supercharges. Note that a higher number of supercharges would restrict the possibilities for the scalar manifolds to a discrete number of symmetric spaces. We first consider vector multiplets in 5 or 4 dimensions with $N = 2$. Vector multiplets in 6 dimensions do not contain scalars. When reducing to 3 dimensions, the vectors become dual to scalars (we can perform duality transformations as we are just considering kinetic terms here, and we can thus restrict to Abelian vectors). Therefore the multiplet in 3 dimensions is dual to a multiplet with only scalars: the hypermultiplet. For hypermultiplets, the spacetime dimension is not really relevant as there are no vectors, and thus the results for hypermultiplets are the same for any dimension. In the picture it is convenient to consider them in 3 dimensions because of the dimensional reduction that we just described. With real scalars in the vector multiplets in 5 dimensions, these geometries are real geometries, those in 4 dimensions are Kählerian, and the hypermultiplets lead to 3 complex structures. Furthermore, we can distinguish between theories that appear in rigid supersymmetry, and those in supergravity. This leads to the overview in the upper part of table 5. The geometries that are related to rigid supersymmetry have been called ‘affine’ in the mathematics literature [40, 41], while

²In principle one could also introduce a metric in the $SU(2)$ part, but as there is no choice for the signature in this sector, this is irrelevant, and we identify $E_i^j = \varepsilon_{ij}$ (with $\varepsilon_{ik} \varepsilon^{jk} = \delta_i^j$).

	$D = 5$ vector multiplets	$D = 4$ vector multiplets	hypermultiplets
rigid (affine)	affine very special real	affine special Kähler	hyperkähler
local (projective)	(projective) very special real	(projective) special Kähler	quaternionic-Kähler

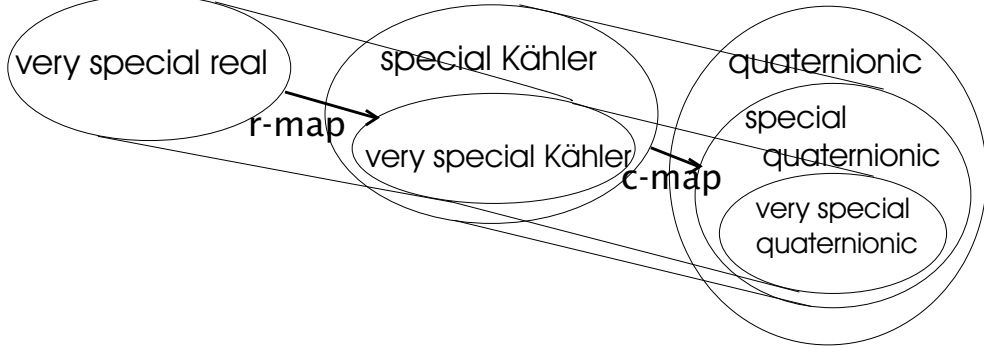


Table 5: *Geometries from supersymmetric theories with 8 real supercharges, and the connections provided by the **r**-map and the **c**-map.*

those for supergravity are called ‘projective’ (and these are the default, in the sense that e.g. special Kähler refers to the geometry that is found in supergravity. The analogous manifolds with 3 complex structures got already a name in the literature.

The name ‘projective’ versus ‘affine’ can be understood from the construction of the manifolds in supergravity using superconformal tensor calculus. We saw already (see table 4) how the real very special manifolds are obtained starting from $(n + 1)$ vector multiplets. Before any gauge fixing, these are just real manifolds with a dilatational invariance. This manifold has therefore a cone structure, with $C_{IJK}h^I h^J h^K$ as the radial coordinate. The physical scalars of the supergravity theory are thus defined modulo this dilatational scaling. The manifolds that occurs in supergravity can thus be seen as a projective space of dimension n .

Similarly, to construct special Kähler geometry, one starts in 4 dimensions with the couplings as they occur in rigid supersymmetry, demanding the presence of a superconformal symmetry. Again, the manifold has a cone structure, and the dilatational gauge condition selects a submanifold at fixed radius. In this case, the superconformal group contains a $U(1)$ invariance and the manifold at fixed radius is a ‘Sasakian manifold’ of dimension $2n + 1$, if this $U(1)$ is not gauged. In conformal supergravity the $U(1)$ is local and eliminates one more scalar. The gauge field of this $U(1)$, which is an auxiliary field in the superconformal tensor calculus (similar to $V_{\mu i}^j$ in table 4), becomes by its field equation the $U(1)$ connection on the Kähler manifold. The final manifold in super-Poincaré has then non-trivial $U(1)$ curvature (and will be a Hodge-Kähler manifold).

The construction for quaternionic manifolds is similar, as has been demonstrated recently in 4 dimensions in [42]. One starts then from hyperkähler cones. The dilatational gauge choice leads to a tri-Sasakian manifold of dimensions $4r + 3$ for ungauged $SU(2)$. The $SU(2)$ gauge fields of the Weyl multiplet get by their field equations the value $V_{\mu i}^j = \partial_\mu q^X \omega_{Xi}^j$, using the $SU(2)$ connection that we had in the previous section. The $SU(2)$ curvature is thus non-zero as required by (2.15).

Dimensional reduction gives a mapping between these manifolds. These mappings have been called the **c**-map (from special Kähler to special quaternionic) [43], and the **r**-map (from very special real to very special Kähler) [44]. They are represented in the lower part of table 5. Dimensional reduction of a manifold in 5 dimensions gives a 4-dimensional theory. But the 4-dimensional theories that can be obtained in this way, are only a subset of all 4-dimensional theories. The table shows the structure in the names given to various classes of manifolds. Very special Kähler manifolds are a subset of all special Kähler manifolds. The quaternionic manifolds that are in the image of the **c**-map are the special quaternionic manifolds, and those in the image of the **c**∩**r**-map are the very special quaternionic manifolds. It is remarkable that nearly all the homogeneous quaternionic manifolds are very special quaternionic manifolds [44] (The only non-special homogeneous quaternionic manifolds are the quaternionic projective spaces).

2.3 Gauging of isometries and the consequences

Having vectors in the vector multiplets (and one graviphoton), these can be used to gauge extra symmetries. The ‘extra’ refers here to the fact that these do not belong to the super-Poincaré or the superconformal algebra. However, at the end, in supergravity, we do not have a strict mathematical algebra of symmetries, but a soft algebra. This means that there are structure functions rather than structure constants. These functions depend on the fields in the theory. In this way the ‘extra’ gauge group can appear in the anticommutator of two supersymmetries with structure functions depending on the fields. When these have a non-zero expectation value, central charges appear. The generators of these gauge group may act also on the fields of the quaternionic manifold.

In supersymmetry, such a gauging has three consequences ([32] gives general properties of gauging in supergravity). The first is that new terms appear in the supersymmetry transformations of the fermions (proportional to a gauge coupling constant). Secondly, the scalar potential is completely determined by the gauging (this is true for theories with 8 supercharges and more). Finally, gauged R -symmetry leads to a cosmological constant. R -symmetry is the symmetry that rotates the supersymmetries and was mentioned in table 1. In $N = 2$, $D = 5$ it is $SU(2)$. We obtain gauged R -symmetry if the extra gauge group contains an $SU(2)$ or a $U(1)$ subgroup thereof. This extra gauge group mixes with the R -symmetry. In the superconformal approach this is due to the gauge fixing of the $SU(2)$ of the superconformal algebra by fixing the scalars in the compensating hypermultiplet, see table 4. The latter in general also transform under the gauge symmetry (see [45] for the structure of the symmetries in the hyperkähler cone). The remaining gauge group is a diagonal subgroup of the superconformal $SU(2)$ and the extra gauge group. The gauge fields of the extra gauge group then gauge the symmetry that acts the supersymmetries (and on the gravitini). In that case we use the terminology ‘gauged supergravity’. We will do this explicitly below for a $U(1) \subset SU(2)$. The formulae can also be used for gauging the full $SU(2)$ as has been done in [46]. Gauged R -symmetry induces a cosmological constant³, which is proportional to the square of the gauge coupling constant (the group theoretical argument was repeated in [4]).

The vectors are in the adjoint of the gauge group, and supersymmetry then implies the same for the scalars h^I in the $(n + 1)$ dimensional space. If the constraint (2.1) is compatible

³To have really a ‘constant’, one still has to assume that there is a solution such that the scalars that determine the value of the potential are constant.

with this, i.e. $C_{L(IJf_K^L)_M} = 0$, then the group can be gauged. It is for these non-Abelian theories that tensor multiplets give extra possibilities [20, 2].

In order that these symmetries can act on the hypermultiplet, the quaternionic manifold should have isometries. Thus, we suppose that there are Killing vectors $K_\alpha^X(q)$ that determine transformations of the scalars q^X , and α denotes the different isometries. In general, only a subset of these can be gauged. We need for each one a gauge vector. Therefore, the appropriate index is I , labeling the vectors (see table 3), and the gauged isometries are determined by $K_I^X(q)$. In quaternionic geometry, the isometries are determined by a triplet of prepotentials $P_I^r(q)$:

$$\mathcal{R}_{XY}^r K_I^Y = D_X P_I^r, \quad D_X P_I^r \equiv \partial_X P_I^r + 2\varepsilon^{rst} \omega_X^s P_I^t. \quad (2.16)$$

These can be solved as well for the Killing vectors, or for the prepotentials:

$$K_I^Z = -\frac{4}{3} \mathcal{R}^{rZX} D_X P_I^r, \quad P_I^r = \frac{1}{2r} \mathcal{R}^{rXY} D_X K_{IY}. \quad (2.17)$$

The latter equation [19] is obviously only true for $r \neq 0$. If there are no physical hypermultiplets ($r = 0$), then P_I^r are just some constants. In the superconformal approach they determine the action of the symmetry on the compensating hypermultiplet. These are the analogues of the Fayet–Iliopoulos terms. For $r > 0$ there is thus no Fayet–Iliopoulos term possible [1]. In rigid supersymmetry, the $SU(2)$ curvatures vanish, and P_I^r are again arbitrary constants or ‘Fayet–Iliopoulos terms’.

The above quantities determine the modified supersymmetries. For that purpose one defines ‘dressed’ Killing vectors and ‘dressed’ prepotentials:

$$P^r \equiv h^I(\phi) P_I^r(q), \quad K^X \equiv h^I(\phi) K_I^X(q). \quad (2.18)$$

These thus depend as well on the scalars of the vector multiplets as on those of the hypermultiplets, but in a well-structured way. The supersymmetry transformations of the fermions are then [2] (bosonic terms only)

$$\begin{aligned} \delta_\epsilon \psi_{\mu i} &= D_\mu(\omega) \epsilon_i + \frac{1}{4\sqrt{6}} i (\gamma_{\mu\nu\rho} - 4g_{\mu\nu} \gamma_\rho) \epsilon_i h_I F^{\nu\rho I} - \frac{1}{\sqrt{6}} i g \gamma_\mu P_i^j \epsilon_j, \\ \delta_\epsilon \lambda_i^x &= -\frac{1}{2} i (\not{D} \phi^x) \epsilon_i + \frac{1}{4} h_I^x \gamma^{\mu\nu} \epsilon_i F_{\mu\nu}^I - g \sqrt{\frac{3}{2}} \epsilon^j \partial_x P_i^j, \\ \delta_\epsilon \zeta^A &= f_X^{Ai} \left[\frac{1}{2} i (\not{D} q^X) \epsilon_i - g \frac{1}{4} \sqrt{6} \epsilon_i K^X \right]. \end{aligned} \quad (2.19)$$

We used here notations of very special geometry:

$$h_I \equiv C_{IJK} h^J h^K, \quad h_I^x = \sqrt{\frac{3}{2}} g^{xy} \partial_y h^I, \quad h_x^I \equiv -\sqrt{\frac{3}{2}} \partial_x h^I. \quad (2.20)$$

Notice that the gauging produced an extra scalar-dependent term for each of the fermions, apart from covariantizations depending on the gauge vectors:

$$D_\mu \epsilon_i = \dots - g A_\mu^I P_{Ii}^j \epsilon_j, \quad D_\mu \phi^x = \dots + g A_\mu^I \sqrt{\frac{3}{2}} h_K f_{JI}^K h^{Jx}, \quad D_\mu q^X = \dots + g A_\mu^I K_I^X. \quad (2.21)$$

Finally, as usual in supersymmetry, the scalar potential is a square of the transformation laws of the fermions. We have here:

$$V = g^2 \left[-4P^r P_r + 3(\partial_x P^r)(\partial_x P_r) + \frac{3}{4} K^X K_X \right]. \quad (2.22)$$

A general form for scalar potentials has been put forward, guaranteeing stability [47, 48, 49]:

$$V = g^2 \left(-6W^2 + \frac{9}{2} g^{\Lambda\Sigma} \partial_\Lambda W \partial_\Sigma W \right), \quad (2.23)$$

where Λ, Σ run over all the scalars, and W is some ‘superpotential’. To make the transition from (2.22), we first split the dressed prepotential in a norm W , which is identified as superpotential, and a phase Q^r :

$$P^r = \sqrt{\frac{3}{2}} W Q^r, \quad Q^r Q^r = 1. \quad (2.24)$$

The phase determines which $U(1)$ subgroup of the $SU(2)$ R -symmetry is gauged. Then it turns out [1] that we can write the potential as (2.23) if the derivative of the phase with respect to the scalars of the vector multiplets is zero:

$$\partial_x Q^r = 0. \quad (2.25)$$

An equivalent condition was found in [50]. This condition is automatic if there are no hypermultiplets (the prepotentials are then constants) or if there are no vector multiplets. However, if one has vector- and hypermultiplets then this is in general only satisfied on a submanifold of the total scalar manifold. We will see below that this condition is required also for BPS solutions of the theory.

3 Smooth solutions: RS, flows and attractors

3.1 BPS conditions

We now look for bosonic solutions with vanishing vectors, a metric of the form (1.1), and preserving some amount of supersymmetry [1]. As the fermions are zero in such solutions, their transformation laws should vanish too. Therefore we investigate the vanishing of (2.19) in this background. The transformation of ψ_5 determines the x^5 -dependence of the Killing spinors $\epsilon^i(x)$, and we can neglect this further. In the transformation of the ψ_μ appears a contribution of the spin connection. It gives the same equation for each $\underline{\mu} = 0, 1, 2, 3$. This equation is still a triplet of equations that can be split in the norm and the phase. These equations are respectively (we use a prime to denote derivatives w.r.t. x^5)

$$gW = \pm \frac{a'}{a}, \quad i\gamma_5 \epsilon_i = \mp Q_i^j \epsilon_j. \quad (3.1)$$

As we defined W to be positive (a norm of a 3-vector), the sign in the first equation depends on the sign of a'/a . The other signs then follow from this one. The second equation is a projection on the preserved supersymmetries. It implies that one half of the supersymmetries survives (4 real supercharges, i.e. $N = 1$ in 4 dimensions).

Note that we have considered only solutions that do not explicitly depend on the coordinates x^μ . On ‘critical points’ (see below), other solutions are possible, doubling the number of preserved supersymmetries (related to the S -supersymmetries in the dual conformal theory).

The transformations of the gauginos can also be split in their norm and phase as $SU(2)$ triplets. The phase gives again (2.25), as we announced already [1]. The equation of the norm and the equation for the hyperinos give a similar condition for all scalars ϕ^Λ [50]:

$$\phi^{\Lambda'} = \mp 3g g^{\Lambda\Sigma} \partial_\Sigma W. \quad (3.2)$$

3.2 Terminology of Renormalization group flow

In the duality between the 5-dimensional theories and 4-dimensional conformal theories, the scalars are dual to coupling constants. The dependence on x^5 is denoted as a ‘flow’. The value of the warp factor a is dual to the energy scale. Therefore, β -functions, i.e. logarithmic derivatives of the coupling constants to the energy, are

$$\beta^\Lambda = a \frac{\partial}{\partial a} \phi^\Lambda = \frac{a}{a'} \phi^{\Lambda'} = -3 \frac{\partial^\Lambda W}{W}, \quad (3.3)$$

where we used (3.1) and (3.2). Critical points ($\beta = 0$) correspond thus to extrema of the superpotential and at these points the scalars are constant. As well for a suitable RS scenario as for a renormalization group flow, the end points of the flow $x^5 = \pm\infty$ should be such critical points.

Whether a critical point is a UV or an IR critical point depends on whether the β -function decreases or increases while passing through its zero. In the first case, the value of the scalars is attracted to this point in the high-energy (large a) regime. In the second case they are attracted to this point in the low-energy (small a) regime. The type of critical point is thus determined by the matrix

$$U_\Sigma^\Lambda \equiv - \left. \frac{\partial \beta^\Lambda}{\partial \phi^\Sigma} \right|_{\beta=0} = 3 \left. \frac{\partial_\Sigma \partial^\Lambda W}{W} \right|_{\partial W=0}. \quad (3.4)$$

Positive eigenvalues imply that the point is a UV attractor for flows in the direction of the corresponding eigenvector, while negative eigenvalues indicate that it can be an IR attractor. The eigenvalues u are the conformal dimensions in the dual theory and the scalar mass is $M^2 = u(u-4) \leq -4$, satisfying the Breitenlohner–Freedman bound [51].

One can prove [19] a general formula for U :

$$\mathcal{U} = \begin{pmatrix} 2\delta_x^y & -\frac{1}{W^2}(\partial_x K^Z)\mathcal{J}_Z^Y \\ \frac{1}{W^2}\mathcal{J}_{XZ}\partial^y K^Z & \frac{3}{2}\delta_X^Y - \frac{1}{W^2}\mathcal{J}_X^Z\mathcal{L}_Z^Y \end{pmatrix}, \quad (3.5)$$

where the first entries are for the vector multiplets and the second for the hypermultiplets, and \mathcal{J} and \mathcal{L} select respectively the $SU(2)$ and $USp(2r)$ part of the dressed gauged isometry, defined by

$$\mathcal{J}_{XY} \equiv 2P^r \mathcal{R}_{XY}^r, \quad D_X K_Y = \mathcal{J}_{XY} + \mathcal{L}_{XY}. \quad (3.6)$$

Note that with only vector multiplets, we have only the upper-left entry of (3.5), and thus only UV attractors [52, 53]. The trace of $\mathcal{J}\mathcal{L}$ is zero, and thus the trace of the full matrix U is $2n + 6r > 0$. Therefore any critical point is UV in some directions.

The ‘attractor equations’, which determine the conditions for critical points, can be written as algebraic equations [1]

$$K^X \equiv h^I K_I^X = 0, \quad P_I^r = h_I h^J P_J^r. \quad (3.7)$$

They have a group-theoretical meaning. The first one says that the ‘dressed symmetry’ at the critical point should be in the stability subgroup of the isometry group. The second has n components, as it is trivial when multiplied with h^I . It says that at the critical point the other n symmetries should have the same $SU(2)$ content as the dressed symmetry.

Also other equations can be understood in a group-theoretical way. The value of the cosmological constant at these points is $-6W^2 = -4P^r P^r$, i.e. determined by the part of the gauging in the $SU(2)$ direction. On the other hand, for an IR critical point, one needs that either the lower-right entry of (3.5) has to become negative, which means that \mathcal{L} has to be large, corresponding to a gauging in the $USp(2r)$ part, or non-diagonal entries should be non-zero, which means that Killing vectors should have parts that are not in the isotropy group of that point.

These results allow to investigate the structure of flows for arbitrary vector- and hypermultiplets in $N = 2$, $D = 5$ supergravity. In [1] couplings of 1 vector and 1 hypermultiplet were considered, and a flow was found that generalizes the IR to UV flow of [54] with two arbitrary parameters. More general models can be investigated easily due to the algebraic nature of the attractor equations and their geometric significance.

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